Solution of Problem 1. *(a) By definition,* (+) *is exact if*

$$D_y f(x,y) = D_x g(x,y),$$

where

$$f(x,y) := y\Big(\frac{1}{\sqrt{xy}} + 1\Big) = \sqrt{\frac{y}{x}} + y,$$
$$g(x,y) := -x\Big(\frac{1}{\sqrt{xy}} - 1\Big) = -\sqrt{\frac{x}{y}} + x,$$

Since

$$D_y f(x,y) = \frac{1}{2}\frac{1}{\sqrt{xy}} + 1 \neq -\frac{1}{2}\frac{1}{\sqrt{xy}} + 1 = D_x g(x,y),$$

it follows that (+) is not exact.

(b) We rewrite the given equation as

$$\mu(x,y)\Big(\sqrt{\frac{y}{x}} + y\Big)dx + \mu(x,y)\Big(-\sqrt{\frac{x}{y}} + x\Big)dy = 0. \tag{1}$$

and let

$$F(x,y) := \mu(x,y)f(x,y) = \mu(x,y)\Big(\sqrt{\frac{y}{x}} + y\Big),$$
$$G(x,y) := \mu(x,y)g(x,y) = \mu(x,y)\Big(-\sqrt{\frac{x}{y}} + x\Big).$$

By definition, (1) *is exact if*

$$D_y F(x,y) = D_x G(x,y),$$

i.e., if

$$\big(D_y\mu(x,y)\big)f(x,y) + \mu(x,y)D_y f(x,y) = \big(D_x\mu(x,y)\big)g(x,y) + \mu(x,y)D_x g(x,y)$$
$$D_y\mu(x,y)\Big(\sqrt{\frac{y}{x}} + y\Big) + \mu(x,y)\Big(\frac{1}{2}\frac{1}{\sqrt{xy}} + 1\Big)$$
$$= D_x\mu(x,y)\Big(-\sqrt{\frac{x}{y}} + x\Big) + \mu(x,y)\Big(-\frac{1}{2}\frac{1}{\sqrt{xy}} + 1\Big)$$
$$\Big(-\sqrt{\frac{x}{y}} + x\Big)D_x\mu(x,y) - \Big(\sqrt{\frac{y}{x}} + y\Big)D_y\mu(x,y) - \frac{1}{\sqrt{xy}}\mu(x,y) = 0$$

(c) For $\mu(x,y) = m(xy)$ *we compute*

$$D_x\mu(x,y) = m'(xy)y,$$
$$D_y\mu(x,y) = m'(xy)x.$$

By (b) we know that (1) *is exact if*

$$\Big(-\sqrt{\frac{x}{y}} + x\Big)D_x\mu(x,y) - \Big(\sqrt{\frac{y}{x}} + y\Big)D_y\mu(x,y) - \frac{1}{\sqrt{xy}}\mu(x,y) = 0,$$

i.e., if

$$-2\sqrt{xy}m'(xy) - \frac{1}{\sqrt{xy}}m(xy) = 0,$$
$$m'(xy) + \frac{1}{2xy}m(xy) = 0.$$

Letting $z := xy$ we get

$$m'(z) + \frac{1}{2z} m(z) = 0,$$
$$\int^z \frac{m'(t)}{m(t)} dt = -\frac{1}{2} \int^z \frac{1}{t} dt$$
$$\Rightarrow m(z) = z^{-\frac{1}{2}}.$$

Going back to the original variables x, y we conclude that

$$m(xy) = \frac{1}{\sqrt{xy}}$$

is an integrating factor for $(+)$.

(d) *By construction, equation*

$$F(x,y)dx + G(x,y)dy = 0$$

with

$$F(x,y) = m(xy)f(x,y) = \frac{1}{x} + \sqrt{\frac{y}{x}},$$
$$G(x,y) = m(xy)g(x,y) = -\frac{1}{y} + \sqrt{\frac{x}{y}},$$

is exact. Thus, its general solution is given by

$$H(x,y) = C = constant,$$

where H satisfies

$$H_x(x,y) = F(x,y),$$
$$H_y(x,y) = G(x,y).$$

We compute

$$H(x,y) = \int F(x,y)dx = \ln x + 2\sqrt{xy} + c(y).$$

Then, from

$$H_y(x,y) = G(x,y),$$

we get

$$c'(y) = -\frac{1}{y} \quad \text{and thus} \quad c(y) = -\ln y.$$

Therefore, the general solution of $(+)$ is

$$\ln \frac{x}{y} + 2\sqrt{xy} = C.$$

Solution of Problem 2. (a) *We compute the first and the second derivatives of $u(x)$:*

$$u'(x) = \alpha e^{\alpha x},$$
$$u''(x) = \alpha^2 e^{\alpha x}.$$

Plugging them into the given equation, we get

$$
\begin{aligned}
(x+1)\alpha^2 e^{\alpha x} + x\alpha e^{\alpha x} - e^{\alpha x} &= 0 \\
\underbrace{e^{\alpha x}\Big(\underbrace{\alpha(\alpha+1)x}_{=0 \text{ for } \alpha=\begin{cases}0\\-1\end{cases}} + \underbrace{(\alpha+1)(\alpha-1)}_{=0 \text{ for } \alpha=\begin{cases}1\\-1\end{cases}} \Big)}_{=0 \text{ for } \alpha=-1} &= 0
\end{aligned}
$$

Thus, $u(x) = e^{-x}$ *is a solution of the given equation.*

(b) Since $x > -1$*, we can devide both of the sides of the given equation by* $(x+1)$*. We get*

$$
y'' + \frac{x}{x+1}y' - \frac{1}{x+1}y = x+1 \tag{2}
$$

We use the suggested ansatz $y(x) = v(x)u(x)$*. First we compute the first and the second derivatives of* y*:*

$$
\begin{aligned}
y'(x) &= v'(x)u(x) + v(x)u'(x), \\
y''(x) &= v''(x)u(x) + 2v'(x)u'(x) + v(x)u''(x).
\end{aligned}
$$

Plugging them into (2)*, we get*

$$
v''u + v'\big(2u' + \frac{x}{x+1}u\big) + v\underbrace{\big(u'' + \frac{x}{x+1}u' - \frac{1}{x+1}u\big)}_{\substack{=0 \\ \text{since } u \text{ solves} \\ \text{the homog. eq.}}} = x+1. \tag{3}
$$

Now we let $w(x) := v'(x)$*. Furthermore, we use* $u(x) = e^{-x}$ *and* $u'(x) = -e^{-x}$*. Thus,* (3) *is equivalent to*

$$
w' + \Big(-2 + \frac{x}{x+1}\Big)w = e^x(x+1). \tag{4}
$$

We solve the homogeneous counterpart of (4)

$$
\begin{aligned}
w' + \Big(-2 + \frac{x}{x+1}\Big)w &= 0 \\
w' &= \frac{x+2}{x+1}w \\
\int^x \frac{w'(t)}{w(t)}dt &= \int^x \frac{t+2}{t+1}dt \\
\Rightarrow w_h(x) &= e^x(x+1)
\end{aligned}
$$

Now, in order to solve the inhomogeneous equation (4)*, we use the variation of constants method. We have to find* $c(x)$ *s.t.*

$$
c'(x)w_h(x) = e^x(x+1),
$$

i.e.,

$$
\begin{aligned}
c'(x)e^x(x+1) &= e^x(x+1) \\
c'(x) &= 1 \\
\Rightarrow c(x) &= x
\end{aligned}
$$

Thus, a particular solution of the inhomogeneous equation is

$$w_p(x) = c(x)w_h(x) = xe^x(x+1)$$

and its general solution is

$$\begin{aligned} w(x) &= Aw_h(x) + w_p(x) \\ &= Aw_h(x) + c(x)w_h(x) \\ &= e^x(x^2 + Ax + x + A) \quad (A = constant) \end{aligned}$$

Going back to v, we get

$$\begin{aligned} v(x) &= \int^x w(t)dt \\ &= \int^x (t^2 + At + t + A)e^t dt \\ &= e^x\left(x^2 + Ax - x + 1\right) + B \quad (B = constant) \end{aligned}$$

Hence,

$$\begin{aligned} y(x) &= v(x)u(x) \\ &= x^2 + Ax - x + 1 + Be^{-x} \end{aligned}$$

We find the constants A and B from the initial conditions

$$\begin{aligned} 1 = y(0) = 1 + B \Rightarrow B = 0 \\ 1 = y'(0) = A - 1 - B \Rightarrow A = 2 \end{aligned}$$

Thus, the solution of the initial value problem is

$$y(x) = x^2 + x + 1.$$

Aufgabe 3

1. die homogene Gleichung $y''' - 8y = 0$ (1)

$x^3 - 8 = 0 = (x-2)(x-(-1+i\sqrt{3}))(x-(-1-i\sqrt{3}))$

Man liest sofort ab:

$$y_h^{(1)}(x) = c_1 e^{2x} + c_2 e^{-x+ix\sqrt{3}} + c_3 e^{-x-ix\sqrt{3}}, \quad c_1, c_2, c_3 \in \mathbb{C}$$

ist die allgemeine Lösung von (1) in komplexer Form

und $y_h^{(2)}(x) = d_1 e^{2x} + e^{-x}(d_2 \cos(\sqrt{3}x) + d_3 \sin(\sqrt{3}x)), \; d_1, d_2, d_3 \in \mathbb{R}$

in reeller Form.

2. Ansatz für die inhomogene Gleichung $y''' - 8y = e^{2x}$

$y_p(x) = Axe^{2x}$, da die Inhomogenität Lösung der homogenen Gleichung ist.

$$\Rightarrow \quad y_p''' - 8y_p = 12A e^{2x} \overset{!}{=} e^{2x} \quad \Rightarrow \quad A = \frac{1}{12}$$

Somit ist die allgemeine Lösung der vorgelegten Gleichung

$$y(x) = y_h^{(1)}(x) + \frac{1}{12} x e^{2x} \qquad \text{(komplexe Form)}$$

$$y(x) = y_h^{(2)}(x) + \frac{1}{12} x e^{2x} \qquad \text{(reelle Form)}$$

Aufgabe 4

Gesucht sind 3 l.u. Lösungen $\vec{x}_1, \vec{x}_2, \vec{x}_3$ des Gleichungssystems

(*) $\quad \dot{\vec{x}}(t) = A\vec{x}(t)$.

Ist $X(t) = [\vec{x}_1(t), \vec{x}_2(t), \vec{x}_3(t)]$, so gilt $e^{tA} = X(t)X(0)^{-1}$.

1. EW und EV von $A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 3 \end{pmatrix}$.

$\det(A - \lambda E) = \underline{(2-\lambda)^2(3-\lambda)}$: EW $\lambda_1 = 2$, $\lambda_2 = 3$

EV zu $\lambda_1 = 2$: $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \vec{x}_1(t) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{2t}$

(Der Eigenraum zu λ_1 ist eindimensional)

EV zu $\lambda_2 = 3$: $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \vec{x}_3(t) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{3t}$

2. Eine weitere Lösung von (*) (zu λ_1) erhält man durch den Ansatz

$$\vec{x}_2(t) = (\vec{a} + \vec{b}t)e^{2t}$$

Einsetzen in die DGL (*) und Koeffizientenvergleich liefern:

$(A - 2E)\vec{b} = \vec{0} \Rightarrow \vec{b} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ (oben)

$(A - 2E)\vec{a} = \vec{b} \Rightarrow \vec{a} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$

$\Rightarrow \vec{x}_2(t) = \begin{pmatrix} t \\ -1 \\ 1 \end{pmatrix} e^{2t}$

also $$X(t) = \begin{pmatrix} e^{2t} & te^{2t} & e^{3t} \\ 0 & -e^{2t} & 0 \\ 0 & e^{2t} & e^{3t} \end{pmatrix}$$

$$\Rightarrow X(0) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \Rightarrow X(0)^{-1} = \begin{pmatrix} 1 & -1 & -1 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\Rightarrow e^{tA} = X(t)X(0)^{-1} = \begin{pmatrix} e^{2t} & e^{2t}(-1-t)+e^{3t} & -e^{2t}+e^{3t} \\ 0 & e^{2t} & 0 \\ 0 & -e^{2t}+e^{3t} & e^{3t} \end{pmatrix}$$