(1) (a) $m\ddot{\vec r} = \vec F = -\vec\nabla U(r) = -\frac{dU(r)}{dr}\vec\nabla r = -\frac{\alpha}{r^3}\vec r \quad , \quad r=\sqrt{x^2+y^2+z^2}$

$\Rightarrow \ddot{\vec r} = -\frac{\alpha}{m}\frac{\vec r}{r^3}$ (1P)

$\vec L = m\vec r\times\dot{\vec r} \Rightarrow \dot{\vec L} = m\underbrace{\dot{\vec r}\times\dot{\vec r}}_{=0} + m\vec r\times\ddot{\vec r} = -\frac{\alpha}{r^3}\vec r\times\vec r = 0$ (1P)

(b) (i) $\frac{\partial}{\partial x_i}U(\vec r) = k\exp\left(\frac{|\vec r-\vec a|}{b}\right)\frac{\partial}{\partial x_i}\frac{|\vec r-\vec a|}{b} = k\exp\left(\frac{|\vec r-\vec a|}{b}\right)\frac{1}{b}\frac{1}{\cancel{2}|\vec r-\vec a|}\cancel{2}(x_i-a_i)$

$\Rightarrow \vec F(\vec r) = -k\exp\left(\frac{|\vec r-\vec a|}{b}\right)\frac{1}{b}\frac{\vec r-\vec a}{|\vec r-\vec a|}$ (1P)

(ii) $\frac{\partial}{\partial x_i}U(\vec r) = -\frac{3}{2}k\frac{1}{|\vec r|^{5/2}}\frac{\partial}{\partial x_i}|\vec r| = -\frac{3}{2}\frac{k}{|\vec r|^{5/2}}\frac{1}{\cancel{2}|\vec r|}\cdot\cancel{2}x_i$

$\Rightarrow \vec F(\vec r) = \frac{3k}{2}\frac{\vec r}{|\vec r|^{7/2}}$ (1P)

(c) (i) $\vec\nabla\times\vec F(\vec r) = \begin{pmatrix}\partial_y F_z - \partial_z F_y\\ \partial_z F_x - \partial_x F_z\\ \partial_x F_y - \partial_y F_x\end{pmatrix} = \begin{pmatrix}\partial_y 0 - \partial_z x\\ \partial_z y - \partial_x 0\\ \partial_x x - \partial_y y\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$

$\Rightarrow \vec F(\vec r)$ ist konservativ (1P)

(ii) $\vec\nabla\times\vec F(\vec r) = \begin{pmatrix}\partial_y(zx) - \partial_z(yz)\\ \partial_z(xy) - \partial_x(zx)\\ \partial_x(yz) - \partial_y(xy)\end{pmatrix} = \begin{pmatrix}0-y\\0-z\\0-x\end{pmatrix} \neq \begin{pmatrix}0\\0\\0\end{pmatrix}$

$\Rightarrow \vec F(\vec r)$ ist nicht konservativ (1P)

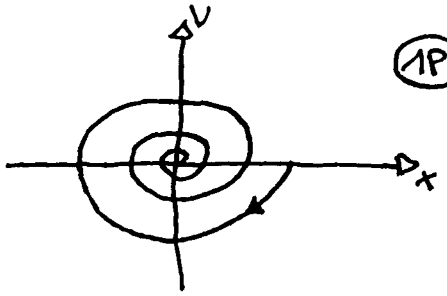
(d) Homogene Lösung: $x_h(t) = Ae^{\sqrt{k}t} + Be^{-\sqrt{k}t}$,

oder: $x_h(t) = C\sinh(\sqrt{k}t) + D\cosh(\sqrt{k}t)$

Partikuläre Lösung: $x_p(t) = -\frac{f}{k}$

Allgemeine Lösung: $x(t) = x_h(t) + x_p(t)$ (2P)

(e) (1P)



2 (a) $\vec F = m\ddot{\vec r} = -m\omega^2\left(a\cos(\omega t)\vec e_x + b\sin(\omega t)\vec e_y\right) = -m\omega^2\vec r$ (1P)

(b) $\vec F(\vec r) = -m\omega^2\vec r\,, \quad \dot{\vec r} = -\omega\left(a\sin(\omega t)\vec e_x - b\cos(\omega t)\vec e_y\right)$

$$\Rightarrow \vec F(\vec r)\cdot\dot{\vec r}\,dt = m\omega^3\left(a^2-b^2\right)\cos(\omega t)\sin(\omega t)\,dt$$

$$\vec r(t=0) = (a,0,0)\,, \quad \vec r\left(t=\tfrac{\pi}{2\omega}\right) = (0,b,0)$$

$$\Rightarrow W = -\int_0^{\frac{\pi}{2\omega}} \vec F(\vec r)\cdot\dot{\vec r}\,dt = m\omega^3\left(b^2-a^2\right)\int_0^{\frac{\pi}{2\omega}} \cos(\omega t)\sin(\omega t)\,dt \quad \text{(1P)}$$

$$= m\omega^3\left(b^2-a^2\right)\left[-\frac{1}{2\omega}\cos^2(\omega t)\right]_0^{\frac{\pi}{2\omega}} = \frac{m\omega^2}{2}\left(b^2-a^2\right) \quad \text{(1P)}$$

<u>Oder:</u> $$W = m\omega^2\int_0^{\pi/2\omega} \vec r\cdot\dot{\vec r}\,dt = m\omega^2\int_0^{\frac{\pi}{2\omega}}\left(\frac{d}{dt}\frac{\vec r^{\,2}}{2}\right)dt$$

$$= m\omega^2\left[\frac{\vec r^{\,2}}{2}\right]_0^{\pi/2\omega} = \frac{m\omega^2}{2}\left(b^2-a^2\right)$$

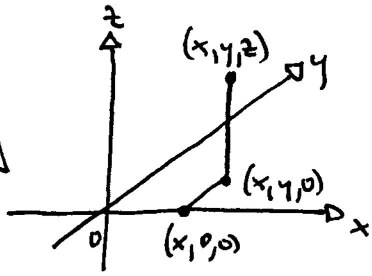
(c) Offensichtlich gilt $U(\vec r) = \frac{m\omega^2}{2}r^2$

Probe: $\frac{\partial}{\partial x_i}U(\vec r) = \frac{m\omega^2}{2}\frac{\partial}{\partial x_i}\sum_{k=1}^{3}x_k^2 = \frac{m\omega^2}{\cancel{2}}\cancel{2}x_i = -F_i$ (1P)

<u>Oder:</u> $$U(\vec r) = -\int_0^{\vec r}\vec F(\vec r\,')\cdot d\vec r\,' = m\omega^2\int_0^{\vec r}\vec r\,'\cdot d\vec r\,'$$

$$= m\omega^2\left[\int_0^x x'dx' + \int_0^y y'dy' + \int_0^z z'dz'\right]$$

$$= \frac{m\omega^2}{2}\left(x^2+y^2+z^2\right) = \frac{m\omega^2}{2}r^2$$



<u>Oder:</u> $-\vec\nabla U(\vec r) = -m\omega^2\vec r \Rightarrow \frac{\partial}{\partial x}U = m\omega^2 x \Rightarrow U = \frac{m\omega^2}{2}x^2 + f(y,z)$

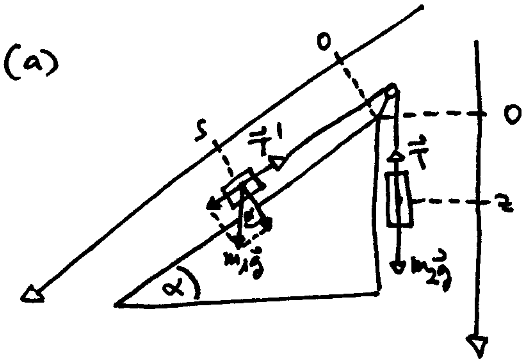
$$\Rightarrow \frac{\partial}{\partial y}U = \frac{\partial}{\partial y}f(y,z) \overset{!}{=} m\omega^2 y \Rightarrow f(y,z) = \frac{m\omega^2}{2}y^2 + g(z)$$

$$\Rightarrow \frac{\partial}{\partial z}U = \frac{\partial}{\partial z}g(z) \overset{!}{=} m\omega^2 z \Rightarrow g(z) = \frac{m\omega^2}{2}z^2 + \text{konst}$$

$$\Rightarrow U(\vec r) = \frac{m\omega^2}{2}\left(x^2+y^2+z^2\right) = \frac{m\omega^2}{2}\vec r^{\,2}$$

$$W = U(0,b,0) - U(a,0,0) = \frac{m\omega^2}{2}\left(b^2-a^2\right) \quad \text{(1P)}$$

③ (a)



$T = |\vec{T}| = |\vec{T}'|$

Seillänge ist konstant

$\Rightarrow s + z = \text{konst}$

$\Rightarrow \ddot{z} = -\ddot{s}$ (1P)

Bewegungsgleichungen: $m_1 g \sin\alpha - T = m_1 \ddot{s}$ (1) (1P)

$m_2 g - T = m_2 \ddot{z} = -m_2 \ddot{s}$ (2) (1P)

$(1) - (2) \Rightarrow m_1 g \sin\alpha - T - m_2 g + T = (m_1 + m_2)\ddot{s}$

$\Rightarrow \ddot{s} = \dfrac{m_1 \sin\alpha - m_2}{m_1 + m_2} g$ (1P)

in (1) $\Rightarrow T = m_2 g + m_2 \ddot{s} = m_2 g \left(1 + \dfrac{m_1 \sin\alpha - m_2}{m_1 + m_2}\right)$

$\Rightarrow T = \dfrac{m_1 m_2}{m_1 + m_2}(1 + \sin\alpha) g$ (1P)

(b) $F_N = m_1 g \cos\alpha$, $F_R = \mu m_1 g \cos\alpha$

Neue Bewegungsgleichungen: $m_1 g \sin\alpha + F_R - T = m_1 \ddot{s}$ (1)

$m_2 g - T = m_2 \ddot{z} = -m_2 \ddot{s}$ (2)

(1P)

$(1) - (2) \Rightarrow m_1 g \sin\alpha - T + \mu m_1 g \cos\alpha - m_2 g + T = (m_1 + m_2)\ddot{s}$

$\Rightarrow \ddot{s} = \dfrac{m_1 \sin\alpha + \mu m_1 \cos\alpha - m_2}{m_1 + m_2} g$ (1P)

in (2) $\Rightarrow T = m_2 g + m_2 \ddot{s} = m_2 g \left(1 + \dfrac{m_1 \sin\alpha + \mu m_1 \cos\alpha - m_2}{m_1 + m_2}\right)$

$\Rightarrow T = \dfrac{m_1 m_2}{m_1 + m_2}(1 + \sin\alpha + \mu \cos\alpha) g$ (1P)

(4) (a) Zentralpotential $\Rightarrow$ Drehimpuls ist erhalten (1P)

(b) $\vec{r} = \rho \vec{e}_\rho \;, \; \dot{\vec{r}} = \dot{\rho}\vec{e}_\rho + \rho\dot{\phi}\vec{e}_\phi$ (1P)

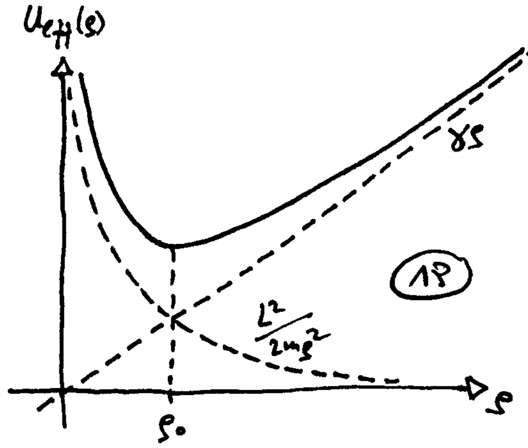
$$\Rightarrow \vec{L} = m\rho\,(\dot{\rho}\underbrace{\vec{e}_\rho \times \vec{e}_\rho}_{=0} + \rho\dot{\phi}\underbrace{\vec{e}_\rho \times \vec{e}_\phi}_{=\vec{e}_z}) = m\rho^2\dot{\phi}\,\vec{e}_z$$ (1P)

(c) $E = \frac{m}{2}\dot{\vec{r}}^2 + U(\vec{r}) = \frac{m}{2}(\dot{\rho}^2 + \rho^2\dot{\phi}^2) + \gamma r$

$$\overset{(b)}{=} \frac{m}{2}\dot{\rho}^2 + \frac{L^2}{2m\rho^2} + \gamma\rho \,,$$ (1P) $\qquad \rho = \sqrt{x^2+y^2+z^2}\Big|_{z=0}$

Kraft ist konservativ (es existiert ein Potential!) $\Rightarrow$ Energieerhaltung (1P)

(d)



$$U'_{eff}(\rho) = -\frac{L^2}{m\rho^3} + \gamma \overset{!}{=} 0$$

$$\Rightarrow \rho_0 = \left(\frac{L^2}{m\gamma}\right)^{1/3}$$ (1P)

$$\Rightarrow L^2 = m\gamma\rho_0^3 \quad (*)$$

$$U_{eff}(\rho_0) = \frac{L^2}{2m\rho_0^2} + \gamma\rho_0 \overset{(*)}{=} \frac{3}{2}\gamma\rho_0 = \frac{3}{2}\gamma\left(\frac{L^2}{m\gamma}\right)^{1/3} = \frac{3}{2}\left(\frac{\gamma^2 L^2}{m}\right)^{1/3}$$ (0,5P)

$$(b) \Rightarrow \dot{\phi} = \frac{L}{m\rho_0^2} \overset{!}{=} \omega_0 \,, \quad L = |\vec{L}| \overset{(*)}{\Rightarrow} \omega_0 = \frac{\sqrt{m\gamma\rho_0^3}}{m\rho_0^2} = \sqrt{\frac{\gamma}{m\rho_0}}$$ (1P)

(e) $$U_{eff}(\rho) = \underbrace{U_{eff}(\rho_0)}_{\text{irrelevante Konstante}} + \underbrace{U'_{eff}(\rho_0)}_{=0,\text{ siehe (d)}}\delta + \frac{1}{2}U''_{eff}(\rho_0)\,\delta^2 + \dots \,, \quad \rho = \rho_0 + \delta$$ (1P)

$$U''_{eff}(\rho) = -\frac{L^2}{m}(-3\rho^{-4}) = \frac{3L^2}{m\rho^4}$$ (1P)

$$\Rightarrow U_{eff}(\rho) = \frac{1}{2}\frac{3L^2}{m}\left(\frac{L^2}{m\gamma}\right)^{-4/3}\delta^2 + \dots = \frac{1}{2}\frac{3L^2}{m\rho_0^4}\delta^2 + \dots$$

$$\overset{(*)}{=} \frac{3}{2}\frac{\gamma}{\rho_0}\delta^2 + \dots \equiv \underbrace{\frac{m\Omega^2}{2}\delta^2 + \dots}_{\text{Potential eines harmonischen Oszillators}}$$ (0,5P)

$$\Rightarrow \Omega = \sqrt{\frac{3\gamma}{m\rho_0}} = \sqrt{3}\,\omega_0$$ (1P)