Moderne Theoretische Physik III SS 2015

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1. Short questions (5 + 10 + 15 + 15 + 5 = 50 Punkte + 15 Bonuspunkte)

(a) The probability distribution for x is just the Boltzamnn distribution

$$p(x) = Ce^{-V(x)/k_BT} \tag{1}$$

with constant C which can be found from normalization

$$C = \frac{1}{a + 2ae^{-V_0/k_BT}}. \tag{2}$$

Thus

$$p_L = \frac{1}{1 + 2e^{-V_0/k_BT}}\,, \qquad p_R = \frac{2e^{-V_0/k_BT}}{1 + 2e^{-V_0/k_BT}}. \tag{3}$$

The probabilities are equal at

$$2e^{-V_0/k_BT} = 1\,, \qquad T = \frac{V_0}{k_B \ln 2}. \tag{4}$$

(b) The partition function of the system reads

$$Z = \sum_{n=1}^{\infty} e^{-E_n/k_BT}. \tag{5}$$

At low temperatures, $k_BT \ll \hbar^2/mL^2$, we can approximate the sum by the first two terms and

$$Z \approx e^{-\frac{\hbar^2\pi^2}{2mL^2k_BT}} \left(1 + e^{-\frac{3\hbar^2\pi^2}{2mL^2k_BT}}\right). \tag{6}$$

We find now the free energy, entropy and heat capacity of the system

$$F = -k_BT\ln Z = \frac{\hbar^2\pi^2}{2mL^2} - k_BT\ln\left(1 + e^{-\frac{3\hbar^2\pi^2}{2mL^2k_BT}}\right) \approx \frac{\hbar^2\pi^2}{2mL^2} - k_BTe^{-\frac{3\hbar^2\pi^2}{2mL^2k_BT}}, \tag{7}$$

$$S = -\partial_T F = k_Be^{-\frac{3\hbar^2\pi^2}{2mL^2k_BT}} + \frac{3\hbar^2\pi^2}{2mL^2T}e^{-\frac{3\hbar^2\pi^2}{2mL^2k_BT}} \approx \frac{3\hbar^2\pi^2}{2mL^2T}e^{-\frac{3\hbar^2\pi^2}{2mL^2k_BT}}, \tag{8}$$

$$c_L = T\partial_T S = -\frac{3\hbar^2\pi^2}{2mL^2T}e^{-\frac{3\hbar^2\pi^2}{2mL^2k_BT}} + \left[\frac{3\hbar^2\pi^2}{2mL^2T}\right]^2 e^{-\frac{3\hbar^2\pi^2}{2mL^2k_BT}} \approx \frac{1}{k_B}\left[\frac{3\hbar^2\pi^2}{2mL^2T}\right]^2 e^{-\frac{3\hbar^2\pi^2}{2mL^2k_BT}}. \tag{9}$$

At high temperatures many terms in the sum are of the same order. We can replace summation by integration. Thus, the partition function reads

$$Z \approx \int_0^\infty dn e^{-\frac{\hbar^2\pi^2n^2}{2mL^2k_BT}} \approx L\sqrt{\frac{mk_BT}{2\pi\hbar^2}}. \tag{10}$$

Free energy, entropy and the heat capacity are

$$F = -Tk_B\ln\left[L\sqrt{\frac{mk_BT}{2\pi\hbar^2}}\right]\,, \tag{11}$$

$$S = k_B\ln\left[L\sqrt{\frac{mk_BT}{2\pi\hbar^2}}\right] + \frac{1}{2}, \tag{12}$$

$$c_L = \frac{k_B}{2}. \tag{13}$$

The value of c_L is in accord with the classical equipartition theorem.

(c) We need first to find the Fermi momentum. In d dimensions we have

$$n = g \int_{|p|<p_F} \frac{d^d p}{(2\pi\hbar)^d} \propto p_F^d, \qquad p_F \propto n^{1/d}. \tag{14}$$

Here g is the spin degeneracy. The Fermi energy

$$E_F = \frac{p_F^2}{2m} \propto n^{2/d}. \tag{15}$$

The heat capacity of a fermi system at low temperatures knows only about the density of states at the Fermi level and is proportional to T in any dimension.

(d) The partition function can be expressed in terms of the transfer matrix as

$$Z = \sum_{\sigma_1,\dots\sigma_N} \mathcal{T}_{\sigma_1,\sigma_2}\mathcal{T}_{\sigma_2,\sigma_3}\dots\mathcal{T}_{\sigma_N\sigma_1} = \mathrm{tr}\mathcal{T}^{\mathrm{N}}. \tag{16}$$

Writing the trace in the basis where $\mathcal{T}$ is diagonal we get

$$Z = \lambda_0^N + \lambda_-^N + \lambda_+^N. \tag{17}$$

In thermodynamic limit only the term with maximal eigenvalue is important and

$$Z = \lambda_+^N. \tag{18}$$

At low temperatures

$$\lambda_+ \approx 2\cosh\beta J + \frac{1}{\cosh\beta J} \approx e^{\beta J} + 3e^{-\beta J}. \tag{19}$$

We thus find

$$F = -NJ - k_B T N \ln\left[1 + 3e^{-2\beta J}\right] \approx -NJ - 3k_B T N e^{-2J/k_B T}, \tag{20}$$

$$S = \frac{N}{T}(2J + k_B T)e^{-2J/k_B T} \approx \frac{2NJ}{T}e^{-2J/k_B T}, \tag{21}$$

$$c = \frac{4J^2 N}{k_B T^2}e^{-2J/k_B T}. \tag{22}$$

(e) We consider the commutation relations

$$\left[a^+, b^+\right] = \left[u\alpha^\dagger + v\beta, u\beta^\dagger + v\alpha\right] = \left[u\alpha^\dagger, v\alpha\right] + \left[v\beta, u\beta^\dagger\right] = -uv + uv = 0 \tag{23}$$

$$\left[a^+, a\right] = \left[u\alpha^\dagger + v\beta, u\alpha + v\beta^\dagger\right] = \left[u\alpha^\dagger, u\alpha\right] + \left[v\beta, v\beta^\dagger\right] = v^2 - u^2 \tag{24}$$

$$\left[b^+, b\right] = \left[u\beta^\dagger + v\alpha, u\beta + v\alpha^\dagger\right] = \left[u\beta^\dagger, u\beta\right] + \left[v\alpha, v\alpha^\dagger\right] = v^2 - u^2 \tag{25}$$

We see that all commutation relations are satisfied provided that $u^2 - v^2$=1.

(f) Differentiating the Landau functional with respect to the order parameter we get the saddle-point equation

$$M(t + 4bM^2 + 6cM^4) = 0. \tag{26}$$

The saddle points are

$$M_0 = 0 \tag{27}$$

$$M_{1,\pm} = \pm\sqrt{\frac{-b - \sqrt{b^2 - 3ct/2}}{3c}}, \tag{28}$$

$$M_{2,\pm} = \pm\sqrt{\frac{-b + \sqrt{b^2 - 3ct/2}}{3c}}. \tag{29}$$

$$\tag{30}$$

They are all real in the vicinity of the point $t = t_c(b) = b^2/2c$. In particular, right at the transition

$$M_{1,\pm} = \pm\sqrt{-\frac{b}{6c}}, \tag{31}$$

$$M_{2,\pm} = \pm\sqrt{-\frac{b}{2c}}. \tag{32}$$

The second derivative of the Landau functional $d^2\mathcal{F}/dM^2$ takes values

$$\left.\frac{d^2\mathcal{F}}{dM^2}\right|_{t=t_c(b),M=0} = t_c b > 0, \tag{33}$$

$$\left.\frac{d^2\mathcal{F}}{dM^2}\right|_{t=t_c(b),M=M_{1,\pm}} = -\frac{b^2}{c} < 0, \tag{34}$$

$$\left.\frac{d^2\mathcal{F}}{dM^2}\right|_{t=t_c(b),M=M_{1,\pm}} = \frac{5b^2}{3c} > 0. \tag{35}$$

Thus in the vicinity of the phase transition M_0 and $M_{2,\pm}$ are the minima of the Landau functional. In this minima the Landau functional takes values

$$\mathcal{F}(M_0) = 0. \tag{36}$$

$$\mathcal{F}(M_{2,\pm}) = M_{2,\pm}^2\left(\frac{t}{2} + bM_{2,\pm}^2 + cM_{2,\pm}^4\right) = M_{2,\pm}^2\frac{6ct + b(-2b + \sqrt{4b^2 - 6ct})}{18c}. \tag{37}$$

In the vicinity of the transition we make the following approximations in Eq. (37):

$$b \approx b_0, \tag{38}$$

$$t \approx \frac{b_0^2}{2c} + \alpha(T - T_c) \equiv \frac{b_0^2}{2c} + \alpha\delta T, \tag{39}$$

$$M_{2,\pm}^2 \approx -\frac{b_0}{2c}. \tag{40}$$

$$\tag{41}$$

We get

$$\mathcal{F}(M_{2,\pm}) \approx -\frac{b_0\alpha\delta T}{4c}. \tag{42}$$

We note that at the point of the phase transition $\mathcal{F}(M_{2,\pm}) = 0 = \mathcal{F}(M_0)$.

We are now ready to give the final results. At $\delta T > 0$ the stable state of the system is $M = M_0 = 0$. The contribution of the magnetic moments to the entropy of the system $\delta S = -\partial_T\mathcal{F}(M_0) = 0$.

At $\delta T < 0$ the stable state is $M = M_{2,\pm} \approx \pm\sqrt{-\frac{b_0}{2c}}$. The contribution of the magnetic moments to the entropy of the system $\delta S = -\partial_T\mathcal{F}(M_{2,\pm}) \approx b_0\alpha/4c$. The magnetization and entropy of the system have jumps at the point of the phase transition.

2. Bose gas with power-law dispersion relation. $(5+5+5+10+10=35$ Punkte)

Let us consider ideal Bose gas in 3 spatial dimensions with the dispersion relation $\epsilon(p)=\epsilon_0(p/p_0)^\alpha$. Here, ϵ_0 and p_0 are constants with dimension of energy and momentum respectively and $\alpha>0$ is a number.
In this exercise you may need the integral

$$\int_0^\infty dx x^\beta e^{-x}=\Gamma(\beta+1)\,,\qquad \beta>-1. \tag{43}$$

Here, $\Gamma(x)$ is the Euler gamma function.

(a) We have for the density of bosons

$$n=\int\frac{d^3p}{(2\pi\hbar)^3}\frac{1}{e^{(\epsilon(p)-\mu)/k_BT}-1}. \tag{44}$$

The number of states with the absolute value of momentum in the interval $(p,p+dp)$ is

$$dn=\frac{4\pi p^2dp}{(2\pi\hbar)^3}=\frac{p^2dp}{2\pi^2\hbar^3}. \tag{45}$$

Thus, the density of states is

$$\nu(\epsilon)=\frac{dn}{d\epsilon}=\frac{p^2}{2\pi^2\hbar^3}\frac{dp}{d\epsilon}=\frac{p_0^3}{2\pi^2\hbar^3}\left(\frac{\epsilon}{\epsilon_0}\right)^{\frac{2}{\alpha}}\frac{d}{d\epsilon}\left(\frac{\epsilon}{\epsilon_0}\right)^{\frac{1}{\alpha}}=\frac{p_0^3}{2\pi^2\hbar^3\alpha\epsilon_0}\left(\frac{\epsilon}{\epsilon_0}\right)^{\frac{3}{\alpha}-1}. \tag{46}$$

Finally

$$n(T,\mu)=\int_0^\infty d\epsilon\nu(\epsilon)\frac{1}{e^{(\epsilon-\mu)/k_BT}-1}=\frac{p_0^3}{2\pi^2\hbar^3\alpha\epsilon_0}\int_0^\infty d\epsilon\left(\frac{\epsilon}{\epsilon_0}\right)^{\frac{3}{\alpha}-1}\frac{1}{e^{(\epsilon-\mu)/k_BT}-1}. \tag{47}$$

(b) At high temperatures $e^{-\mu/k_BT}\gg1$. Thus

$$n(\mu,T)\approx\frac{p_0^3}{2\pi^2\hbar^3\alpha\epsilon_0}\int_0^\infty d\epsilon\left(\frac{\epsilon}{\epsilon_0}\right)^{\frac{3}{\alpha}-1}\frac{1}{e^{(\epsilon-\mu)/k_BT}}=\frac{p_0^3\Gamma[3/\alpha]}{2\pi^2\hbar^3\alpha}\left(\frac{k_BT}{\epsilon_0}\right)^{3/\alpha}e^{\mu/k_BT}. \tag{48}$$

Thus,

$$\mu(n,T)=-k_BT\ln\left[\frac{p_0^3\Gamma[3/\alpha]}{2\pi^2\hbar^3\alpha n}\left(\frac{k_BT}{\epsilon_0}\right)^{3/\alpha}\right]. \tag{49}$$

(c) The Ω-potential an ideal Bose gas is given by

$$\Omega(T,V,\mu)=k_BTV\int\frac{d^3p}{(2\pi\hbar)^3}\ln\left[1-e^{(\mu-\epsilon(p))/k_BT}\right]=k_BTV\int_0^\infty d\epsilon\nu(\epsilon)\ln\left[1-e^{(\mu-\epsilon)/k_BT}\right]. \tag{50}$$

At $e^{\mu/k_BT}\ll1$ we can expand the logarithm. We get (we take into account expression for the density of states derived in 2a)

$$\Omega(T,V,\mu)=-k_BTV\frac{p_0^3}{2\pi^2\hbar^3\alpha\epsilon_0}\int_0^\infty d\epsilon\left(\frac{\epsilon}{\epsilon_0}\right)^{\frac{3}{\alpha}-1}e^{(\mu-\epsilon)/k_BT}$$
$$=-k_BTV\frac{p_0^3\Gamma[3/\alpha]}{2\pi^2\hbar^3\alpha}\left(\frac{k_BT}{\epsilon_0}\right)^{\frac{3}{\alpha}}e^{\mu/k_BT}=-V\frac{p_0^3\epsilon_0\Gamma[3/\alpha]}{2\pi^2\hbar^3\alpha}\left(\frac{k_BT}{\epsilon_0}\right)^{\frac{3}{\alpha}+1}e^{\mu/k_BT}. \tag{51}$$

(d) The entropy reads now

$$S(T,V,\mu) = -\partial_T \Omega = V \frac{p_0^3 \epsilon_0 \Gamma[3/\alpha]}{2\pi^2 \hbar^3 \alpha} e^{\mu/k_B T} \left(\frac{k_B T}{\epsilon_0}\right)^{\frac{3}{\alpha}+1} \left[\frac{1}{T}\left(\frac{3}{\alpha}+1\right) - \frac{\mu}{k_B T^2}\right]. \tag{52}$$

We substitute expression for μ derived in 2b to get

$$S(T,V,N) = k_B V n \left[\left(\frac{3}{\alpha}+1\right) - \frac{\mu}{k_B T}\right] = \left(\frac{3}{\alpha}+1\right) k_B N + k_B N \ln \left[\frac{p_0^3 \Gamma[3/\alpha]}{2\pi^2 \hbar^3 \alpha n} \left(\frac{k_B T}{\epsilon_0}\right)^{3/\alpha}\right]. \tag{53}$$

The heat capacity is

$$c_V = T \partial_T S = \frac{3}{\alpha} N k_B T. \tag{54}$$

This is in agreement with the equipartition theorem for $\alpha = 2$.

(e) We come back to the expression for the number of bosons in the system

$$n(T,\mu) = \frac{p_0^3}{2\pi^2 \hbar^3 \alpha \epsilon_0} \int_0^\infty d\epsilon \left(\frac{\epsilon}{\epsilon_0}\right)^{\frac{3}{\alpha}-1} \frac{1}{e^{(\epsilon-\mu)/k_B T} - 1}. \tag{55}$$

For $\alpha < 3$ the integral converges at $\epsilon = 0$ even for $\mu = 0$. Thus, there is a maximal number of bosons that can be accommodated in the single particle states with non-zero momentum. If the density of the system is larger than

$$n_c(T) = \frac{p_0^3}{2\pi^2 \hbar^3 \alpha \epsilon_0} \int_0^\infty d\epsilon \left(\frac{\epsilon}{\epsilon_0}\right)^{\frac{3}{\alpha}-1} \frac{1}{e^{\epsilon/k_B T} - 1}, \tag{56}$$

macroscopic number of particles condenses into the state $p = 0$. This is the Bose condensation. On the other hand, for $\alpha \geq 3$ the integral in Eq. (55) diverges at small energies if $\mu = 0$. Thus, there is no Bose condensation.

To estimate the critical temperature we switch in Eq. (56) to dimensionless integration variable $y = \epsilon/k_B T$. We find

$$n_c(T) = \frac{p_0^3}{2\pi^2 \hbar^3 \alpha} \left(\frac{k_B T}{\epsilon_0}\right)^{\frac{3}{\alpha}} \int_0^\infty dy y^{\frac{3}{\alpha}-1} \frac{1}{e^y - 1} \propto T^{3/\alpha}. \tag{57}$$

Thus,

$$T_c \propto n^{\alpha/3}. \tag{58}$$

3. Particle in magnetic field and Langevin equation

(5 + 10 = 15 Punkte + 10 + 10 = 20 Bonuspunkte)

Consider a particle with charge e and mass m moving on a plane in perpendicular magnetic field B. The particle also experiences friction force $-m\gamma\vec{v} = -m\gamma(v_x, v_y)$ from the surrounding media and random Langevin force $\vec{\xi}(t) = (\xi_x(t), \xi_y(t))$. The noise $\xi(t)$ is characterised by the correlator

$$\langle \xi_\alpha(t) \xi_\beta(t') \rangle = q \delta_{\alpha\beta} \delta(t - t'). \tag{59}$$

(a) The equations of motion for our particle include the Lorentz force, the friction force and the Langevin forces

$$m\dot{v}_x + \frac{eB}{c} v_y + m\gamma v_x = \xi_x, \tag{60}$$

$$m\dot{v}_y - \frac{eB}{c} v_x + m\gamma v_y = \xi_y. \tag{61}$$

To solve the Langevin equations it is convenient to introduce

$$a = v_x + iv_y\,, \qquad a^\star = v_x - iv_y. \tag{62}$$

We have

$$m\dot{a} - i\frac{eB}{c}a + \gamma m a = \xi_x + i\xi_y, \tag{63}$$

$$m\dot{a}^\star + i\frac{eB}{c}a^\star + \gamma m a^\star = \xi_x - i\xi_y. \tag{64}$$

(b) We discuss no the solution of the first equation. The solution for $a^\star$ can be obtained by complex conjugation.
In the absence of Langevin forces we have

$$a = Ce^{i\Omega t - \gamma t}, \qquad \Omega = \frac{eB}{mc}. \tag{65}$$

We now allow C to vary in time and get

$$m\dot{C} = (\xi_x(t) + i\xi_y(t))e^{-i\Omega t + \gamma t}. \tag{66}$$

Thus,

$$C(t) = C_0 + \frac{1}{m}\int_0^t d\tau(\xi_x(\tau) + i\xi_y(\tau))e^{-i\Omega\tau + \gamma\tau} \tag{67}$$

$$a(t) = \left[C_0 + \frac{1}{m}\int_0^t d\tau(\xi_x(\tau) + i\xi_y(\tau))e^{-i\Omega\tau+\gamma\tau}\right]e^{i\Omega t - \gamma t}. \tag{68}$$

Due to our initial conditions $C_0 = 0$ and we get finally

$$a(t) = \frac{e^{i\Omega t - \gamma t}}{m}\int_0^t d\tau(\xi_x(\tau) + i\xi_y(\tau))e^{-i\Omega\tau+\gamma\tau}. \tag{69}$$

We can now find v_x and v_y by taking real and imaginary parts of $a(t)$

$$\begin{aligned} a(t) = v_x + iv_y &= \frac{1}{m}\int_0^t d\tau(\xi_x(\tau) + i\xi_y(\tau))e^{(i\Omega-\gamma)(t-\tau)} \\ &= \frac{1}{m}\int_0^t d\tau e^{-\gamma(t-\tau)}(\xi_x(\tau) + i\xi_y(\tau))\left[\cos\Omega(t-\tau) + i\sin\Omega(t-\tau)\right], \\ &= \frac{1}{m}\int_0^t d\tau e^{-\gamma(t-\tau)}\left[\xi_x(\tau)\cos\Omega(t-\tau) - \xi_y(\tau)\sin\Omega(t-\tau)\right] \\ &\quad + \frac{i}{m}\int_0^t d\tau e^{-\gamma(t-\tau)}\left[\xi_x(\tau)\sin\Omega(t-\tau) + xi_y(\tau)\cos\Omega(t-\tau)\right]. \end{aligned} \tag{70}$$

Thus,

$$v_x(t) = \frac{1}{m}\int_0^t d\tau e^{-\gamma(t-\tau)}\left[\xi_x(\tau)\cos\Omega(t-\tau) - \xi_y(\tau)\sin\Omega(t-\tau)\right] \tag{71}$$

$$v_y(t) = \frac{1}{m}\int_0^t d\tau e^{-\gamma(t-\tau)}\left[\xi_x(\tau)\sin\Omega(t-\tau) + xi_y(\tau)\cos\Omega(t-\tau)\right]. \tag{72}$$

(c) We compute now the correlation functions one by one.

$$\langle v_x(t)v_x(t')\rangle = \frac{1}{m^2}\int_0^t d\tau \int_0^{t'} d\tau' e^{-\gamma(t-\tau)-\gamma(t'-\tau')}\langle[\xi_x(\tau)\cos\Omega(t-\tau) - \xi_y(\tau)\sin\Omega(t-\tau)]$$
$$\times \left[\xi_x(\tau')\cos\Omega(t'-\tau') - \xi_y(\tau')\sin\Omega(t'-\tau')\right]\rangle$$
$$= \frac{q}{m^2}\int_0^{\min(t,t')} d\tau e^{-\gamma t-\gamma t'-2\gamma\tau}\left[\cos\Omega(t-\tau)\cos\Omega(t'-\tau) + \sin\Omega(t-\tau)\sin\Omega(t'-\tau)\right]$$
$$= \frac{q}{m^2}\cos\Omega(t-t')\int_0^{\min(t,t')} d\tau e^{-\gamma t-\gamma t'-2\gamma\tau}$$
$$= \frac{q}{2m^2\gamma}\cos\Omega(t-t')e^{-\gamma(t+t')}\left(e^{2\gamma\min(t,t')} - 1\right) = \frac{q}{2m^2\gamma}\cos\Omega(t-t')\left(e^{-\gamma|t-t'|} - e^{-\gamma(t+t')}\right). \tag{73}$$

In an analogous way we get

$$\langle v_y(t)v_y(t')\rangle = \frac{q}{2m^2\gamma}\cos\Omega(t-t')\left(e^{-\gamma|t-t'|} - e^{-\gamma(t+t')}\right), \tag{74}$$

$$\langle v_x(t)v_y(t')\rangle = \frac{q}{2m^2\gamma}\sin\Omega(t-t')\left(e^{-\gamma|t-t'|} - e^{-\gamma(t+t')}\right). \tag{75}$$

(d) In the limit $t = t' \to \infty$ we get

$$\langle v_x(t)v_x(t)\rangle = \langle v_y(t)v_y(t)\rangle = \frac{q}{2m^2\gamma}, \qquad \langle\frac{m\vec{v}^2}{2}\rangle = \frac{q}{2m\gamma}. \tag{76}$$

Comparing it to the equipartition theorem we get

$$q = 2m\gamma T. \tag{77}$$